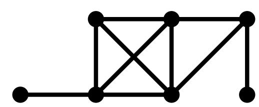
MA 111, Topic 4: Graph Theory

Our last topic in this course is called **Graph Theory**. This is the mathematics of connections, associations, and relationships.

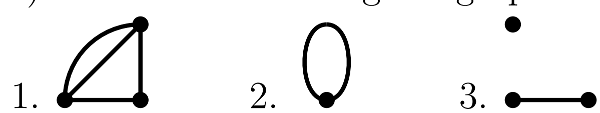
Definition 1. A **Graph** is a set of points called **Vertices** (singular **Vertex**) and lines called **Edges** that connect some of the vertices. Graphs are used to understand links between objects and people.

Example 2 (Graph Introduction)**.** Below is a graph with 7 vertices and 10 edges.



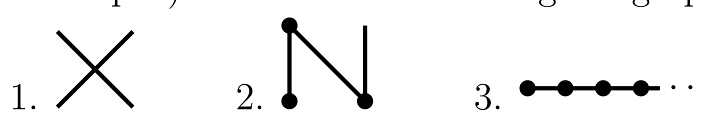
Notice there is no vertex at the overlap between two edges. Vertices will/must always be clearly distinguished!

Example 3 (Graphs)**.** All of the following are graphs.



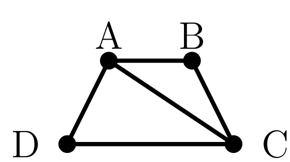
Multi-edges, loops, and two or more pieces are all allowed.

Example 4 (Not Graphs)**.** None of the following are graphs.



No vertices scenarios, edges without ending vertices, and infinite vertices will not be allowed here.

REMEMBER, graphs are supposed to indicate connections between things. There are many different ways to describe these connections. Consider the graph below:



Each Vertex is labeled with a capital letter. This is enough to describe the graph in its entirety.

- The **Vertex Set** is just a collection given by the labels we put on the vertices. In the example above the Vertex Set is described by $\{A, B, C, D\}$.

- The **Edge Set** is the collection of the edges, described using the endpoint vertices of each edge. In the example above the Edge Set is described by $\{AB, AC, AD, BC, CD\}$.

Example 5 (Spring Classes 1)**.** A group of college friends discuss which classes they will be taking next semester. They make a chart of their names and all their classes below. In the chart a "✓" means that person listed in that row is taking the class listed in that column.

	Econ	**Geography**	**History**	**Math**	**Philosophy**
Aaron	✓	✓		✓	✓
Ben		✓	✓		✓
Claire	✓				
Danielle			✓	✓	

- Make a graph where the vertices are "People" and edges connect "Possible Study Partners (for any class)".

- How many vertices are in your graph? How many edges?

Example 6 (Spring Classes 2)**.** A group of college friends discuss which classes they will be taking next semester. They make a chart of their names and all their classes below. In the chart a "✓" means that person listed in that row is taking the class listed in that column.

	Econ	**Geography**	**History**	**Math**	**Philosophy**
Aaron	✓	✓		✓	✓
Ben		✓	✓		✓
Claire	✓				
Danielle			✓	✓	

- Make a new graph with an edge for every class that could be studied together.

- How many vertices are in your graph? How many edges?

Example 7 (Spring Classes 3)**.** Some people decide to change up their schedules.

	Econ	**Geography**	**History**	**Math**	**Philosophy**
Aaron	✓	✓			
Ben		✓	✓		
Claire				✓	✓
Danielle				✓	✓

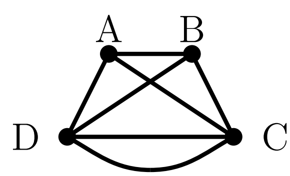
- Make a graph with an edge for every class that could be studied together.

- How many vertices are in your graph? How many edges?

Definition 8 (Basic Graph Definitions/Notation 1)**.** There are a few more vocabulary words that often get used with with graphs.

- We usually use the letter v to indicate the number of vertices of a graph. This is just a count of things that appear in the vertex set. Sometimes we call this number the **Order** of the graph.
- We usually use the letter e to indicate the number of edges of a graph. This is just a count of things that appear in the edge set.
- A graph is **Connected** if it is all one piece. Stated in another way, from any starting vertex we can trace along the edges of the graph to any other vertex *WITHOUT ANY JUMPS!*

Example 9 (Example of Basic Graph Definitions/Notation 1)**.** In the graph below, we have $v = 4$.



In other words, the order of the graph is 4. Notice this is the same as the number of things in the vertex set $\{A, B, C, D\}$.

This graph also has 7 edges. So $e = 7$. This value is the same as the number of things in the edge set

$$\{AB, AC, AD, BC, BD, CD, CD\}.$$

This graph is all one piece, so it is *connected*.

Example 10 (Spring Classes 4)**.** A few more friends (Erica, Fiona, and Greg) want to compare upcoming class schedules with everyone before. The friends decide that it is too much trouble to make a new chart. Instead they list connections whenever two people have at least one class in common. They get the following list:

$$\{AB, AC, AD, AE, BD, BE, BG, CD, CE, DE, EG\}$$

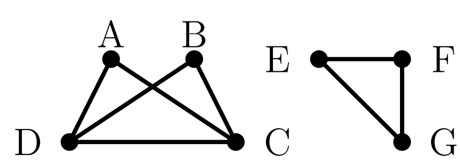
- Draw a graph that represents the classes in common between friends A through G.

- What is the order of your graph? Is your graph connected?

Definition 11 (Basic Graph Definitions/Notation 2)**.** Here are two more definitions that we will use:

- A graph is **Disconnected/Unconnected** if it has more than one piece. In other words, this means it is NOT ONE PIECE.
- The connected pieces of a graph are called the **Components**. We will usually use the letter c to describe the number of components a graph has. A connected graph has $c = 1$. Any graph with $c > 1$ must be disconnected.

Example 12 (Example of Basic Graph Definitions/Notation 2)**.** This graph is disconnected with $v = 7$, $e = 8$, and $c = 2$.



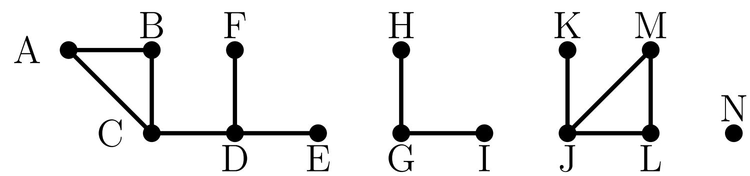
Example 13 (Spring Classes 5)**.** Consider the edge set below:

$$\{AB, AC, AD, CD, EF, FG, HI, HJ, HL, LM, LN, MN\}$$

- Draw a graph with vertices A through N and edges listed above.

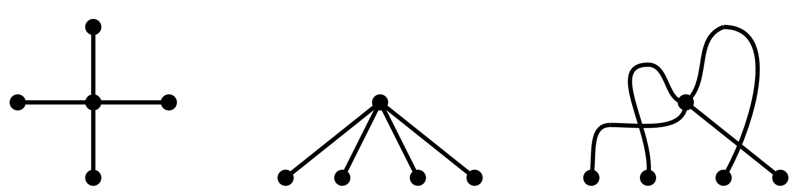
- How many components does your graph have?

- Is your graph the same as the graph below?



Isomorphic Graphs

Some graphs are the "same" even though they aren't drawn in the same way. The graphs below are all the same, because they represent identical kinds of connections.



Definition 14 (Isomorphic Graphs)**.** Graphs that have the same number of vertices and identical connections (but may look different) are said to be **isomorphic**.

Given two graphs, it is often really hard to tell if they ARE isomorphic, but usually easier to see if they ARE NOT isomorphic. Here is our first idea to help tell if two graphs are isomorphic.

Theorem (Isomorphic Graphs Theorem 1)**.** Suppose we have two graphs. In the first graph there are v_1 vertices and e_1 edges. In the second graph there are v_2 vertices and e_2 edges. Then in order for the two graphs to be isomorphic we must have:

- $v_1 = v_2$
- $e_1 = e_2$

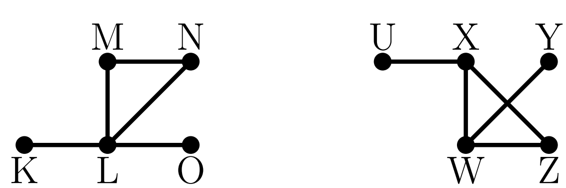
In words, isomorphic graphs must have the same number of vertices and edges.

It is important to note that just having $v_1 = v_2$ and $e_1 = e_2$ is **NOT** a guarantee that two graphs will be isomorphic.

Example 15 (Isomorphic or Not Isomorphic 1)**.** Consider the following:

- Are the two graphs below isomorphic?

- Are the two graphs below isomorphic?



Example 16 (Isomorphic or Not Isomorphic 2)**.** Answer the following:

- Draw the graph with vertices A, B, C, D and edge set

$$\{AB, AC, AD, BC\}$$

- Is your graph isomorphic to one of the graphs below?

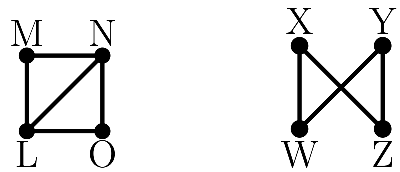
- Can you relabel the vertices of either pictured graph to match your graph?

Example 17 (Isomorphic or Not Isomorphic 3)**.** Answer the following:

- Draw the graph with vertices A, B, C, D and edge set

$$\{AB, AC, AD, BC, BD\}$$

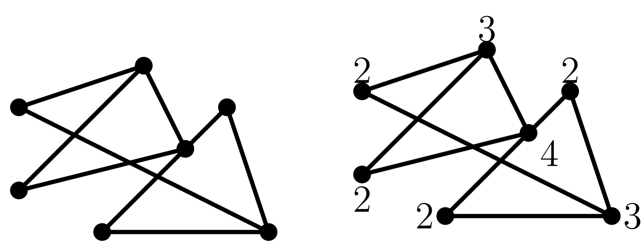
- Is your graph isomorphic to one of the graphs below?



- Can you relabel the vertices of either pictured graph to match your graph?

Definition 18 (Degree of Vertex)**.** The **degree of a vertex** is the number of edges attached to that vertex.

Example 19 (Degree of Each Vertex Shown)**.** The right graph below has the vertices labeled with their degrees.



We can use the idea of degree of a vertex to help us better understand when two graphs might be isomorphic.

Theorem (Isomorphic Graphs Theorem 2)**.** Suppose we have two graphs where each graph has the same number of vertices, $v_1 = v_2 = n$. Write the degrees of each vertex (with repeats) in ascending order for Graph 1. This gives a list of numbers that we can represent generally as $d_1,\ d_2,\ d_3, \ldots,\ d_n$.

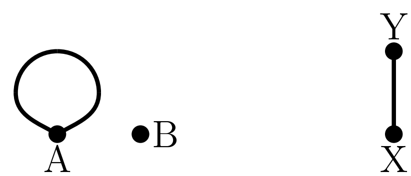
If the two graphs are isomorphic then when listing the degrees of Graph 2 in ascending order, we get the exact same list as above.

In short, **ISOMORPHIC GRAPHS HAVE THE SAME DEGREE LISTS**.

More useful though, **IF THE DEGREE LISTS ARE DIFFERENT, THE TWO GRAPHS ARE NOT ISOMORPHIC**.

There are more sophisticated ways to determine if two graphs are isomorphic, but generally this is a VERY HARD question to resolve.

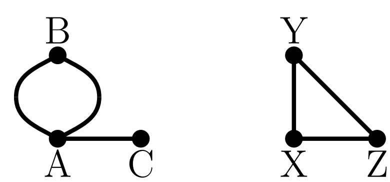
Example 20 (Counting Connections 1)**.** Consider the following graphs below:



- What is a reason(s) for why these graphs could be isomorphic?

- What is a reason(s) for why these graphs could *NOT* be isomorphic?

Example 21 (Counting Connections 2)**.** Consider the following graphs below:



- What is a reason(s) for why these graphs could be isomorphic?

- What is a reason(s) for why these graphs could *NOT* be isomorphic?

Example 22 (Counting Connections 3)**.** Consider the following graphs below:

- What is a reason(s) for why these graphs could be isomorphic?

- What is a reason(s) for why these graphs could *NOT* be isomorphic?

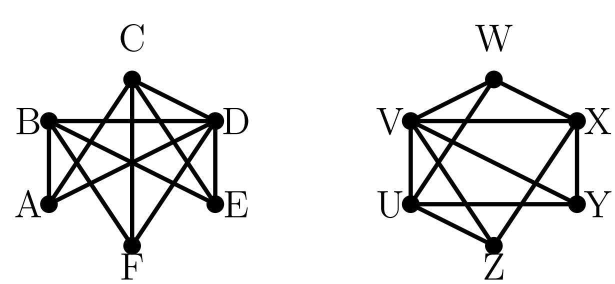
The "Determining If Two Graphs Are Isomorphic Theorems 1 & 2" are mostly useful for showing that two graphs are **NOT** isomorphic.

Definition 23 (Graph Isomorphism)**.** If two graphs are isomorphic then there is a **Graph Isomorphism** that describes how they are the same. In practice this is:

- A relabeling of the vertices of Graph 1 so that each corresponds to the "same" vertex of Graph 2;
- This relabeling is done so that any edge of Graph 1 has a corresponding edge of Graph 2 under the new labels.

To determine a graph isomorphism, a really good place to start is to find the degrees of the vertices of BOTH graphs.

Example 24 (Describing a Graph Isomorphism)**.** Here is a method to describe a graph isomorphism between two graphs.



- Step 1: List the degrees, in ascending order, of both graphs:

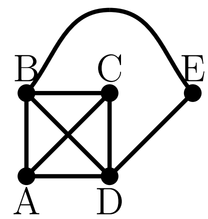
$$\begin{array}{ll} \text{Left Graph} & \overset{3}{A},\ \overset{3}{E},\ \overset{3}{F},\ \overset{4}{B},\ \overset{4}{C},\ \overset{5}{D} \\ \text{Right Graph} & \overset{3}{W},\ \overset{3}{Y},\ \overset{3}{Z},\ \overset{4}{U},\ \overset{4}{X},\ \overset{5}{V} \end{array}$$

- Step 2: Understand the connections of the vertices:
  - In the Left Graph, vertices A, E, and F all connect with B, C, and D. These are their only connections.
  - In the Right Graph, vertices W, Y, and Z all connect with U, V, and X. These are their only connections.
  - In both graphs, Vertices of degree 4 connect with the one Vertex of Degree 5.
- Step 3: Define the isomorphism:
  - Because of the types of connections, any vertex of degree 3 in the Left Graph is like any vertex of degree 3 in the Right Graph. The same is true for the vertices of degree 4.
  - $A \leftrightarrow W$, $E \leftrightarrow Y$, $F \leftrightarrow Z$, $B \leftrightarrow U$, $C \leftrightarrow X$, $D \leftrightarrow V$.
  - We can check by redrawing the Right Graph with vertices in the positions given by the isomorphism.

Example 25 (Using Vertex Degrees 1)**.** Answer the following:

- Make a graph with 5 vertices labeled with the letters A, B, C, D, and E.
- Include 8 edges connecting the vertices. You decide which connections to make!

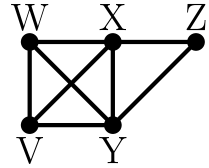
- Is your graph isomorphic to this one?



Example 26 (Using Vertex Degrees 2)**.** Answer the following:

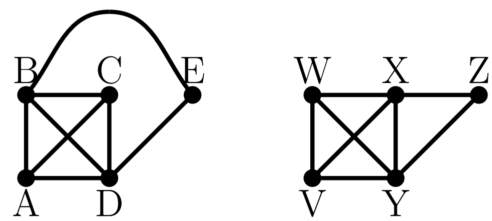
- Make another graph with 5 vertices labeled with the letters V, W, X, Y, and Z.
- Connect the vertices so that the degrees are 2, 3, 3, 4, 4

- Is your graph isomorphic to this one?

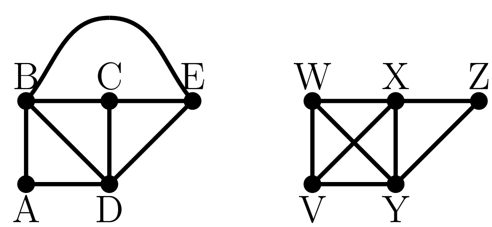


Example 27 (Using Vertex Degrees 3)**.** Answer the following:

- Are the two graphs below isomorphic?



- Are the two graphs below isomorphic?



Example 28 (Compare With a Neighbor 1)**.** Previously you made a graph with 5 vertices labeled with the letters letters A, B, C, D, and E, connected by 8 edges.

- Compare your graph with a neighbor's graph. Are your two graphs isomorphic?
  - If not isomorphic, how can you tell?
  - If they are isomorphic, find a correspondence between vertices.

Example 29 (Compare With a Neighbor 2)**.** Previously you made a graph with: 5 vertices labeled with the letters V, W, X, Y, and Z, connected so the degrees are 2, 3, 3, 4, 4.

- Compare your graph with a neighbor's graph. Are your two graphs isomorphic?
  - If not isomorphic, how can you tell?
  - If they are isomorphic, find a correspondence between vertices.

Theorem (Sum of Degrees of Vertices Theorem)**.** Suppose a graph has n vertices with degrees $d_1, d_2, d_3, \ldots, d_n$. Add together all degrees to get a new number $d_1 + d_2 + d_3 + \ldots + d_n = D_v$. Then $D_v = 2e$. In words, for any graph the sum of the degrees of the vertices equals twice the number of edges. Stated in a slightly different way, $D_v = 2e$ says that D_v is <u>ALWAYS</u> an even number.

Example 30 (Using the Sum of Degrees of Vertices Theorem)**.** It is impossible to make a graph with $v = 6$ where the vertices have degrees 1, 2, 2, 3, 3, 4. This is because the sum of the degrees D_v is

$$D_v = 1 + 2 + 2 + 3 + 3 + 4 = 15$$

D_v is always an even number but 15 is odd!

Example 31 (Using the Sum of the Degrees of Vertices Formula 1)**.** Consider the following scenarios:

- A graph has 4 vertices with degrees 0, 0, 0, and 0. What does this graph look like?

- A graph has 1 vertex with degree 2. What does this graph look like?

- A graph has 4 vertices with degrees 2, 3, 3, and 4. How many edges are there?

- A graph has 4 vertices with degrees 2, 2, 2, and 4. Can you say what this graph looks like?

Example 32 (Using the Sum of the Degrees of Vertices Formula 2)**.**
Consider the following scenarios:

- Is it possible to have a graph with vertices of degrees:
  1 and 1?

- Is it possible to have a graph with vertices of degrees:
  1 and 2?

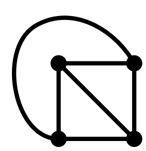
- Is it possible to have a graph with vertices of degrees:
  1, 1, 2, 3?

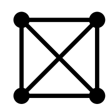
- Is it possible to have a graph with vertices of degrees:
  1, 1, 2, 3, 3?

Planar Graphs

Definition 33 (Planar Graph)**.** A graph is **Planar** if can be drawn in such a way that its edges do not cross. To determine if a graph is planar we have to consider isomorphic versions of the graph.

Example 34 (Using Isomorphisms to Make Planar)**.** The graph on the left is definitely planar (edges do not cross). Even though the graph to the right does not appear to be planar,**it is because it is isomorphic to the graph to the left!**



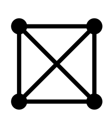


Definition 35 (Faces of a Planar Graphs)**.** In any planar graph, drawn with no intersections, the edges divide the planes into different regions. The regions enclosed by the planar graph are called **interior faces** of the graph. The region surrounding the planar graph is called the **exterior (or infinite) face** of the graph.

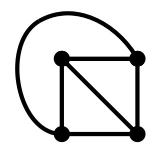
When we say **faces** of the graph we mean **BOTH** the interior AND the exterior faces. We usually denote the number of faces of a planar graph by f.

BEFORE YOU COUNT FACES, IT IS VERY IMPORTANT TO FIRST DRAW A PLANAR GRAPH SO THAT NO EDGES CROSS!

Example 36 (Counting Faces of a Planar Graphs)**.** It is really tempting to try to count the faces in the graph below:



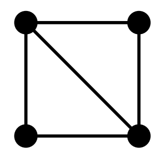
Remember, the idea of faces only applies to Planar Graphs drawn with no edge crossings. Instead we need to redraw the graph to get an isomorphic version:



In the bottom version it is easy to see there are *3 Interior Faces and 1 Exterior Face for a total of 4 Faces.*

Definition 37 (Degree of a Face)**.** For a planar graph drawn without edges crossing, the number of edges bordering a particular face is called the **degree of the face**.

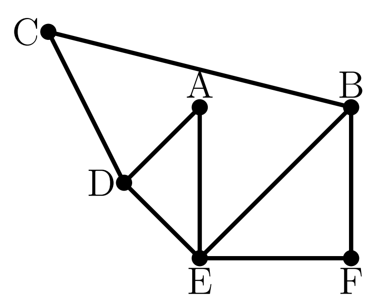
In the graph below



each of the two interior faces have degree 3. The infinite (exterior) face has degree 4. Adding up degrees gives a result that is curiously similar to one before.

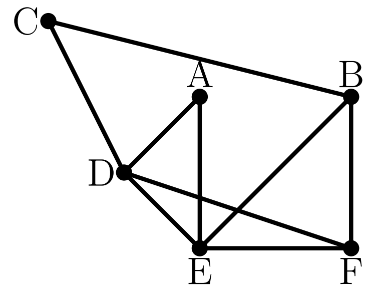
Theorem (Sum of the Degrees For Faces)**.** In any planar graph, the sum of the degrees of all faces is equal to twice the number of edges. In symbols, $D_f = 2e$.

Example 38 (Counting Faces and Degrees 1)**.** Consider the graph:



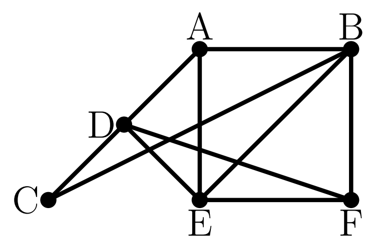
- How many faces does this graph have?
- What is the degree of each face?
- List v, e, and f for this graph.

Example 39 (Counting Faces and Degrees 2)**.** Consider the graph:



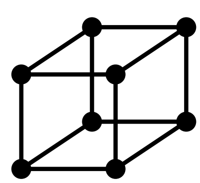
- How many faces does this graph have?
- What is the degree of each face?
- List v, e, and f for this graph.

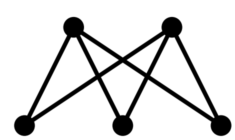
Example 40 (Counting Faces and Degrees 3)**.** Consider the graph:



- How many faces does this graph have?
- What is the degree of each face?
- List v, e, and f for this graph.

Here are a couple more graphs that don't look planar at first. Can you count the faces?





	Left Graph (Cube)	**Right Graph ($K_{3,2}$)**
Vertices v		
Edges e		
Faces f		

Theorem (Euler's Formula)**.** Take any connected planar graph drawn with no intersecting edges.
Let v be the number of vertices in the graph.

Let e be the number of edges in the graph.

Let f be the number of faces in the graph.

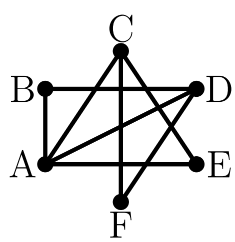
$$\text{Then } v - e + f = 2.$$

Check the planar graphs on the previous slide:

- Cube $\quad v - e + f = 8 - 12 + 6 = 2$
- $K_{3,2}$ $\quad v - e + f = 5 - 6 + 3 = 2$

Let's see what Euler's Formula can do for us!

Example 41 (Using Euler's Formula 1)**.** Consider the graph:



- What is v?
- What is e?
- If the graph is planar, what must f be?
- Redraw the graph with vertices/edges moved around so the graph appears planar.

Example 42 (Using Euler's Formula 2)**.** Answer the following:

- A connected planar graph has 8 vertices and 12 edges. How many faces are there?

- A connected planar graph has 6 vertices and 4 faces. How many edges are there?

- A connected planar graph has 8 vertices with degrees:
  1, 1, 2, 2, 3, 3, 4, 4. How many edges are there?

- A connected planar graph has 8 vertices with degrees:
  1, 1, 2, 2, 3, 3, 4, 4. How many faces are there?

Example 43 (Using Euler's Formula 3)**.** Answer the following:

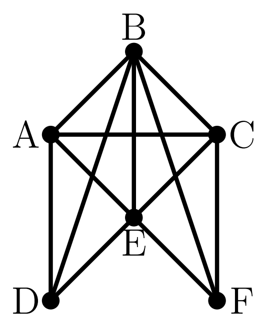
- A connected planar graph has 6 faces and 12 edges. How many vertices are there?

- A connected planar graph has 4 faces with degrees:
  3, 3, 4, 4. How many edges are there?

- A connected planar graph has 4 faces with degrees:
  3, 3, 4, 4. How many vertices are there?

- Draw a connected planar graph with 4 faces with degrees: 3, 3, 4, 4. Is there only one such isomorphic graph?

Example 44 (Using Euler's Formula 4)**.** Consider the graph:



- What is v? What is e?
- Though it doesn't look it, this is graph is planar. What is f?

We can combine the following theorems to answer questions about planar graphs.

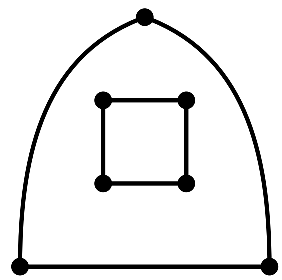
Theorem (Sum of the Degrees For Vertices)**.** In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Theorem (Sum of the Degrees For Faces)**.** In any planar graph, the sum of the degrees of all faces is equal to twice the number of edges.

Theorem (Euler's Formula)**.** For a connected planar graph with vertices v, edges e, and faces f, the following must hold:

$$v - e + f = 2.$$

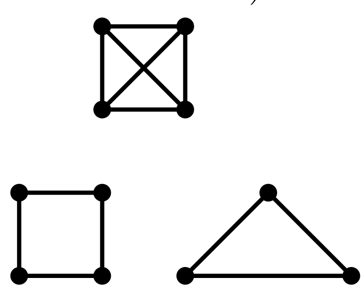
Note that Euler's Formula only applies to *connected* graphs, i.e., graphs that have *one component* where we write $c = 1$. See if you can extend Euler's Formula for $c = 2$ in the graph below!



Theorem (Euler's General Formula)**.** For any planar graph with vertices v, edges e, faces f and components c, the following must hold:

$$v - e + f - c = \_\_.$$

Example 45 (Another Euler's Formula 1)**.** Consider the graph:



- Find v, e, and f for this graph.
- Find c for this graph.
- Combine c, e, f, and v to check your new Euler's Formula!

Example 46 (Another Euler's Formula 2)**.** Answer the following:

- A connected planar graph has vertices whose degrees are 3, 3, 4, 4, 5, 6, 7. How many vertices are there?

- A connected planar graph has vertices whose degrees are 3, 3, 4, 4, 5, 6, 7. How many edges are there?

- A connected planar graph has vertices whose degrees are 3, 3, 4, 4, 5, 6, 7. How many faces are there?

- A disconnected planar graph with $c = 2$ has vertices whose degrees are 3, 3, 4, 4, 5, 6, 7. How many faces are there?

- A disconnected planar graph with $c = 4$ has vertices whose degrees are 3, 4, 5, 6, 7, 8, 9. How many faces are there?

Coloring the Vertices of a Graph

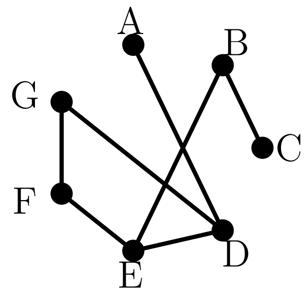
We are going to assign colors to vertices of a graph in order to do some applications. Here are some rules for how this must be done:

- **Every vertex must be colored.**
- **Any two vertices that are connected by an edge must have a different color.**

This leads to our next definition.

Definition 47 (n-Colorable and Chromatic Number)**.** A graph is n**-colorable** if it can be colored with n colors so that adjacent vertices (those sharing an edge) do not have the same color. The smallest possible number of colors needed to color the vertices is called the **Chromatic Number** of the graph.

Example 48 (Vertex Coloring 1)**.** Consider the graph below.



- Is this graph planar?
- Can you color the vertices of the graph using 4 colors?
- Can you color the vertices of the graph using 3 colors?
- What is the Chromatic Number of the graph?

There's something REALLY obvious about coloring vertices of a graph, but let's talk about it anyway. Any time there is an edge between two vertices

we will need at least TWO colors to for the vertices.

The following is not a mind-blowing result, but it's a start!

Theorem (Chromatic Number 1 Theorem)**.** A graph has Chromatic Number 1 exactly when there are NO EDGES. In other words, the graph must be entirely vertices.

Note that in our examples so far:

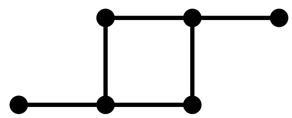
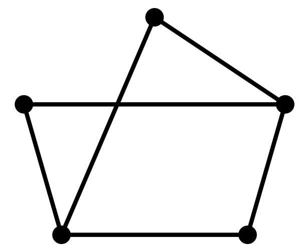
- Some of the graphs are similar, but they have different chromatic numbers.
- There is a connection between certain chromatic numbers and the way in which you can make a "round trip" in a graph.

Definition 49 (Circuit)**.** A **circuit** of a graph is a route through adjacent vertices that starts at a certain vertex and ends at the same vertex as the route started.

So what's the connection between Chromatic Number of a graph and circuits? Well, it has something to do with Chromatic Number 2 and the kind of circuits we can find.

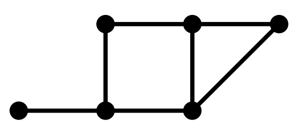
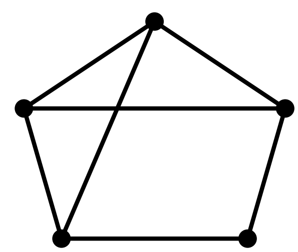
Theorem (Chromatic Number 2 Theorem)**.** A graph has Chromatic Number 2 exactly when there are NO circuits with an odd number of vertices.

The two graphs below have Chromatic Number 2 because of the Theorem above.

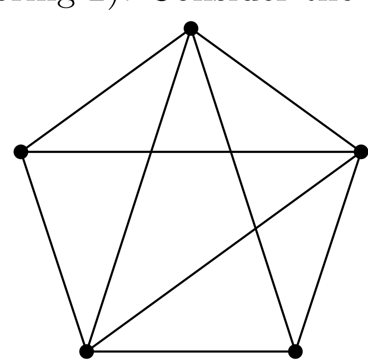
The circuit for the graph on the left has 4 vertices. Circuits for the graph on the right have either 4 or 6 vertices.
Notice that the Chromatic Number 2 Theorem tells us when a graph is NOT 2-colorable as well.

Example 50 (Graphs that are not 2-colorable)**.** The two graphs below have do not Chromatic Number 2:

In these graphs we can easily find circuits with 3 or 5 vertices. So the **Chromatic Number 2 Theorem** says that these graphs DO NOT have chromatic number 2.

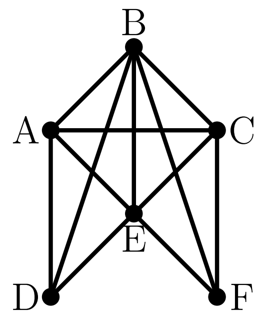
Example 51 (Vertex Coloring 2)**.** Consider the graph below.



- How do you know this graph is not 1-colorable?

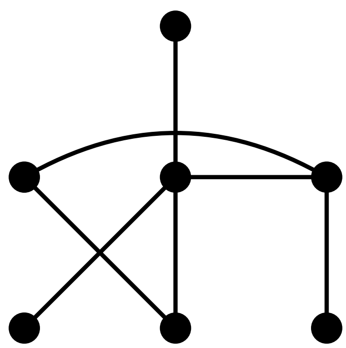
- How do you know this graph is not 2-colorable?
- Find the Chromatic Number.

Example 52 (Vertex Coloring 3)**.** Consider the graph below.



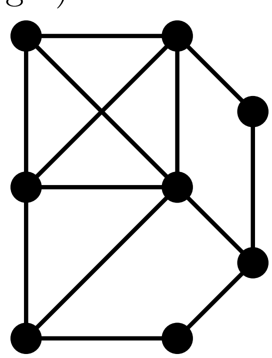
- How do you know this graph is not 2-colorable?
- Is the graph 3-colorable?
- Find the Chromatic Number.

Example 53 (Vertex Coloring 4)**.** Consider the graph below.



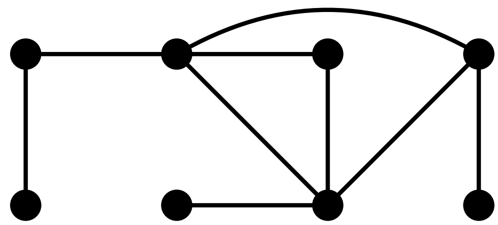
- Find the Chromatic Number.

Example 54 (Vertex Coloring 5)**.** Consider the graph below.



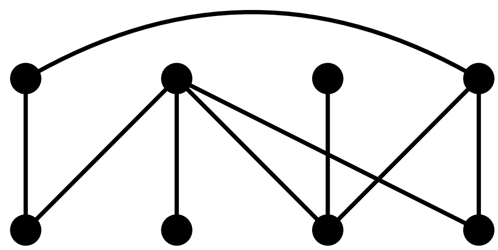
- Find the Chromatic Number.

Example 55 (Vertex Coloring 5.5)**.** Consider the graph below.



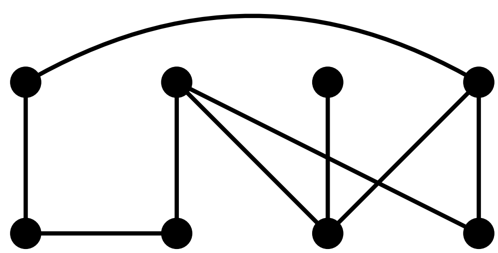
- Find the Chromatic Number.

Example 56 (Vertex Coloring 6)**.** Consider the graph below.



- Find the Chromatic Number.

Example 57 (Vertex Coloring 7)**.** Consider the graph below.



- Find the Chromatic Number.

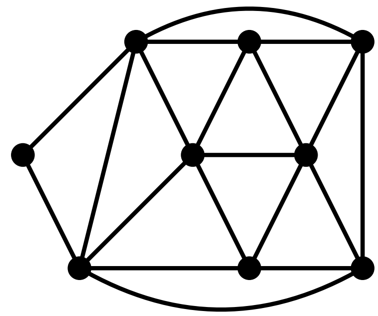
There is a reason we care about planar graphs.

- Every map can be converted into a planar graph.
- We can use graph coloring for some applications, but these colorings originally came from coloring our maps.
- There is something amazing about planar graphs that was first conjectured in the 19th century and took over 100 years to prove (finally in 1976 by Haken & Appel):

Theorem (The Four-Color Theorem)**.** Every planar graph is 4-colorable. In other words, if a graph is planar then it has Chromatic Number 1, 2, 3, or 4.

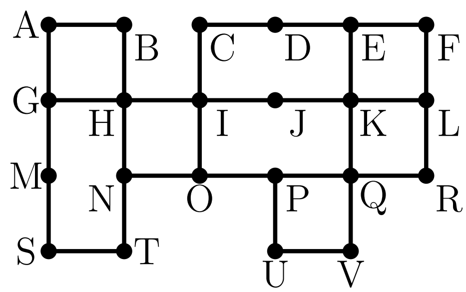
In particular, you can color **ANY** map with 4 or fewer colors.

Example 58 (Vertex Coloring 8)**.** Consider the graph below.



- Before trying to color vertices, what is the highest value that the Chromatic Number could be?
- Find the Chromatic Number.

Example 59 (Vertex Coloring 9)**.** Consider the graph below.



- Before trying to color vertices, what is the highest value that the Chromatic Number could be?
- Is the graph 2-colorable?
- Find the Chromatic Number.

Modeling With Graphs

We like to make ideas of graphs useful. Here are some important things to remember about all graphs.

- Graphs always represent how "things" are connected or related. Using graphs means recognizing the things and what kinds of connections are being described.
- Generally speaking, **Vertices** always represent the "things" being connected.
- **Edges** always represent the connections, but these depend on the "things" you choose to represent your vertices.

- A good rule of thumb: If it is hard to figure out the edges, then you probably made the vertices too hard.

Here are some specific of how graphs model human interactions.

Networks: This is a scenario where people (things) are connected to other people in some way. This could be through friendship (like on *Facebook*), starring in a movie or writing a paper together.

- **Vertices** are people.
- **Edges** between people whenever there is an association, like friendship.

Scheduling: This is a scenario where people (things) are connected to other people, but through conflict. We want the graph to represent the conflict in some way.

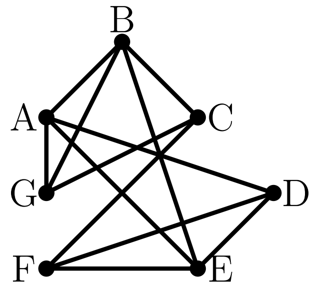
- **Vertices** (usually) represent people.
- **Edges** between people whenever there is a conflict.

Example 60 (Graph Modeling 1)**.** The math department is having difficulties scheduling courses A–G because of limited room availability. In the chart below, an "X" means two courses cannot be scheduled at the same time.

	A	B	C	D	E	F	G
A		X		X	X		X
B	X		X		X		X
C		X				X	X
D	X				X	X	
E	X	X		X		X	
F			X	X	X		
G	X	X	X				

- Make a graph with vertices A–G. Make an edge between vertices if the corresponding courses cannot be scheduled at the same time.

Example 61 (Graph Modeling 2)**.** The math department is having difficulties scheduling courses A–G because of limited room availability. The graph below represents the scheduling conflicts.



- Find the Chromatic Number of the graph above.
- What does the Chromatic Number tell you?

Example 62 (Graph Modeling 3: Scheduling Campus Clubs)**.** In this table, an X means that there is at least one student in both clubs. For scheduling we need to first find the number of meeting times needed for all students to be able to attend.

	Ski	Book	SCRAP	GSA	Game	Bible	Dem.	Rep.
Ski	–	X						X
Book	X	–	X	X	X			
SCRAP		X	–	X		X		X
GSA		X	X	–	X	X	X	
Game		X		X	–	X	X	
Bible			X	X	X	–	X	X
Dem.				X	X	X	–	
Rep.	X		X			X		–

- The graph we make should have vertices representing clubs. How should we make the edges?
- Make the graph for this chart and find its Chromatic number.

- What does this tell you about scheduling?

Example 63 (Graph Modeling 4: Scheduling Sorority Sisters)**.** Seven sorority sisters (Alison, Brittney, Caitlynn, Dana, Elaine, Fiona, and Genna) must meet to plan an annual retreat weekend. Some sisters don't get along, as indicated by an **X** in the chart:

	A	B	C	D	E	F	G
A	–			X			
B		–	X		X		
C		X	–				
D	X			–	X		X
E		X		X	–	X	
F					X	–	X
G				X		X	–

- Make a graph that will help you determine the number of scheduling times needed.

- How many meeting times are needed?

Here are some specific of how graphs model maps and places.

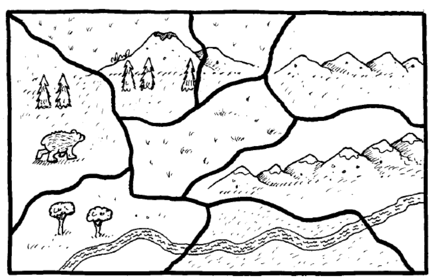
Maps or Places: This is a scenario where the things are states, regions, rooms, or some other locations. The connection between locations is some kind of proximity measure, either a border or passage of some kind.

- **Vertices** are physical or geographical locations.
- **Edges** whenever there is a way to pass from one location to an adjacent location.

Traffic or Movement: This is a scenario where the things are traffic or paths of objects. Connections between traffic/paths represents possible accidents.

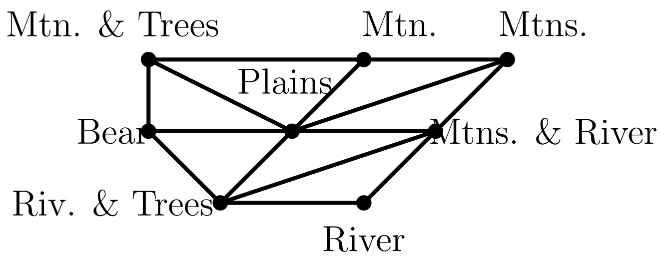
- **Vertices** are flows of traffic or paths.
- **Edges** whenever flow along one path can affect flow along another.

Example 64 (Graph Modeling 5: The Bear Trap!)**.** A bear (pictured) is followed closely by a bear hunter (not pictured). Whenever the bear crosses from one region into another the hunter lays a number of bear traps along that border.



- Make a graph with vertices **EACH REGION** the bear can visit. The edges of your graph should represent possible border crossings.

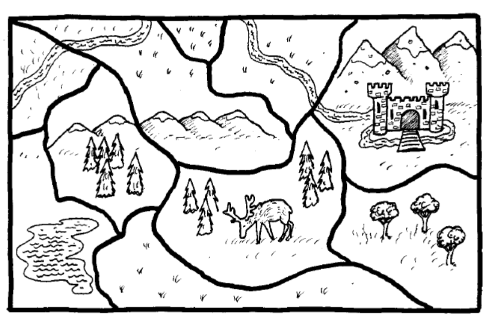
Example 65 (Graph Modeling 5: The Bear Trap!)**.** The bear really likes to visit regions by crossing different borders, so the bear trapper is a big problem for the bear. Fortunately, the bear knows some graph theory and makes the graph below:



- Can the bear make a round-trip to every region without crossing the same border twice?

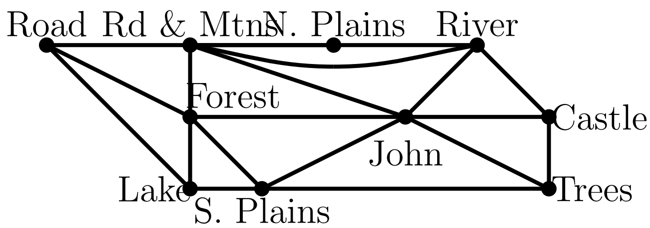
- Can the bear make a round-trip, crossing every border, without having to cross any border twice?

Example 66 (Graph Modeling 6: Deer John)**.** John the Deer (pictured) is friends with the bear from the previous question. He learns graph theory from the bear and uses it to understand the regions where he (John the Deer) likes to hang.



- Make a graph that uses regions and borders for vertices and edges.

Example 67 (Graph Modeling 6: Deer John)**.** John the Deer (pictured) is friends with the bear from the previous question. He learns graph theory from the bear and uses it to understand the regions where he (John the Deer) likes to hang.



- Can John the deer make a round-trip, crossing every border, without having to cross any border twice?

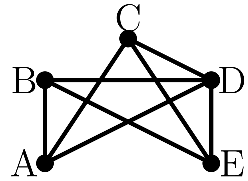
- Can John the deer make *ANY* trip, crossing every border, without having to cross any border twice?

Euler Circuits and Euler Paths

Definition 68 (Circuits and Paths)**.** Circuits have been defined earlier, but here is their definition again along with the similar definition for a path.

- A **circuit** of a graph is a route through adjacent vertices that starts at a vertex and ends back at the beginning.
- A **path or trail** of a graph is a route through adjacent vertices that starts at a certain vertex and ends at a different vertex.

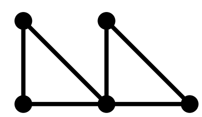
Example 69 (Circuits and Paths)**.** Consider the graph:



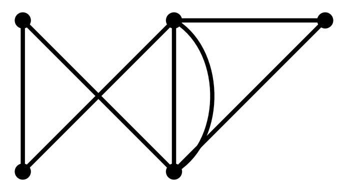
Definition 70 (Eulerian Graphs)**.** A route through a graph that covers **every** edge *EXACTLY ONCE* with an ending vertex that is the same as the starting vertex is called an **Euler Circuit**. A graph is said to be **Eulerian** if there is an **Euler circuit**.

We'll eventually figure out which graphs can have Euler Circuits. For now here's an example

Example 71 (Euler Circuit)**.** Consider the graph:

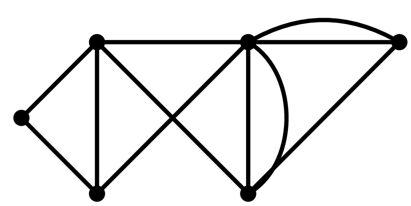


Example 72 (Travel Every Edge 1)**.** Consider the graph:



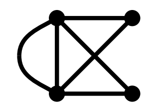
- Can you find a *circuit* that hits every edge without repeating any edges?
- What are the degrees of the vertices of this graph?

Example 73 (Travel Every Edge 2)**.** Consider the graph:



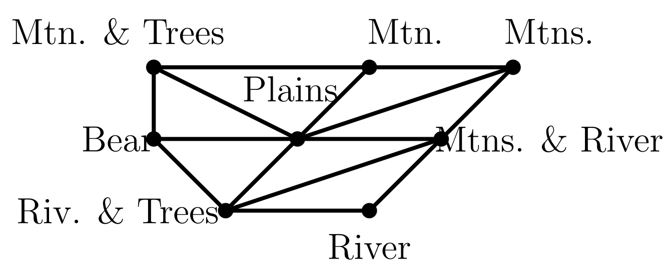
- Can you find a circuit that hits every edge without repeating any edges?
- What are the degrees of the vertices of this graph?

Example 74 (Travel Every Edge 3)**.** Consider the graphs:



- In the graphs above, can you find a circuit that hits every edge without repeating any edges?
- Find the degrees of the vertices of the graphs above!
- Make a conjecture (or guess) about how to tell when you can get circuits to travel over every edge.

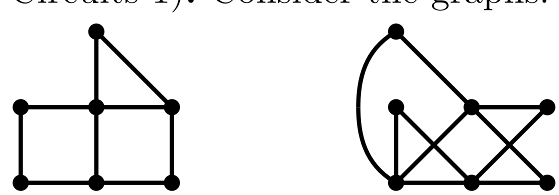
Example 75 (Travel Every Edge 4)**.** Consider the graph:



- What are the degrees of the vertices of this graph?

- Do you think you can you find a circuit that hits every edge without repeating any edges?

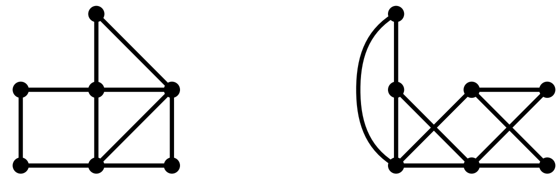
Example 76 (Euler Circuits 1)**.** Consider the graphs:



- Find the degrees of the vertices of the graphs above!

- Which of the graphs has an Euler Circuit?
- Where can your Euler Circuit begin and end?

Example 77 (Euler Circuits 2)**.** Consider the graphs:



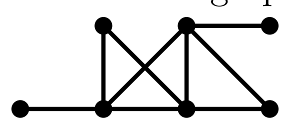
- Find the degrees of the vertices of the graphs above!

- Which of the graphs has an Euler Circuit?
- Where can your Euler Circuit begin and end?

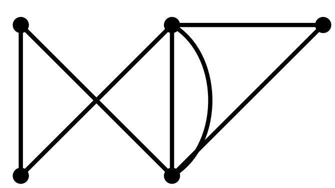
Definition 78 (Semi-Eulerian Graphs)**.** A route through a graph that covers **every** edge *EXACTLY* once with an ending vertex that is different than the starting vertex is called an **Euler Path** (also called an **Euler Trail**). A graph is said to be **Semi-Eulerian** if there is an **Euler Path**.

An Euler Path is a little different. We have to start and end at different vertices

Example 79 (Euler Path)**.** Consider the graph:

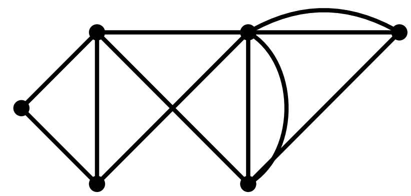


Example 80 (Travel Every Edge 5)**.** Consider the graph:



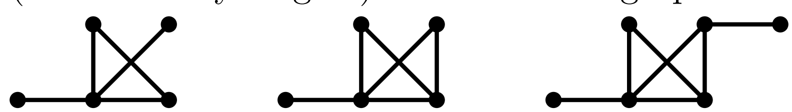
- Can you find a *path* (not a round trip!) that hits every edge without repeating any edges?
- What are the degrees of the vertices of this graph?

Example 81 (Travel Every Edge 6)**.** Consider the graph:



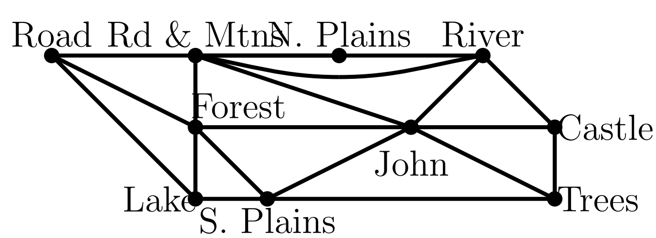
- Can you find a *path* (not a round trip!) that hits every edge without repeating any edges?
- What are the degrees of the vertices of this graph?

Example 82 (Travel Every Edge 7)**.** Consider the graphs:



- In the graphs above, can you find a path that hits every edge without repeating any edges?
- Find the degrees of the vertices of the graphs above!
- Make a conjecture (or guess) about how to tell when you can get path to travel over every edge.

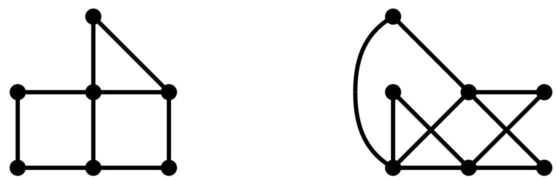
Example 83 (Travel Every Edge 8)**.** Consider the graph:



- What are the degrees of the vertices of this graph?

- Do you think you can you find a path that hits every edge without repeating any edges?

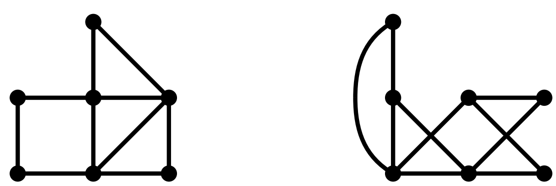
Example 84 (Euler Paths 1)**.** Consider the graphs:



- Find the degrees of the vertices of the graphs above!

- Which of the graphs has an Euler Path?
- Where must your Euler Path(s) begin and end?

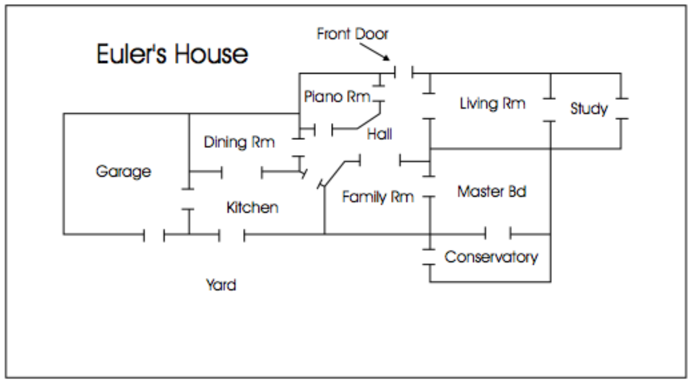
Example 85 (Euler Path 2)**.** Consider the graphs:



- Find the degrees of the vertices of the graphs above!

- Which of the graphs has an Euler Path?
- Where must your Euler Path(s) begin and end?

Example 86 (Euler's House)**.** Euler likes to think about his house, whose blueprint is below:



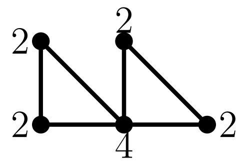
- Can Euler travel through every doorway exactly once to make an Euler Circuit?

- Can Euler travel through every doorway exactly once to make an Euler Path?

We are ready for our first **BIG** Theorem in Graph Theory. These are the results that started Graph Theory about 300 years ago!

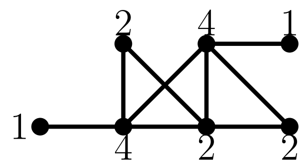
Theorem (Euler Circuit Theorem)**.** A connected graph has an Euler Circuit exactly when every vertex has even degree.

Example 87 (Degrees and Euler Circuits)**.** Notice all even degrees in an example from before.

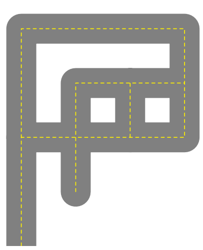


Theorem (Euler Path Theorem)**.** A connected graph contains an Euler Path exactly when all but two vertices have even degree. The Euler Path will begin and end at the vertices with odd degree.

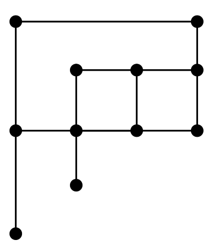
Example 88 (Degrees and Euler Paths)**.** Notice all but two even degrees in an example from before.



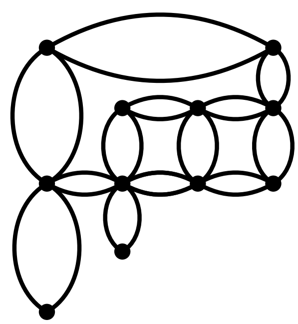
WARNING: Euler's Theorems only tell us when Euler Circuits or Euler Paths exist. They do not tell us how to find them.
We know from practical experience that there should always be a way to make Euler Circuits and Paths *AS LONG AS WE ARE OKAY SOMETIMES DOUBLING BACK*.



Street Map



No Euler Circuit

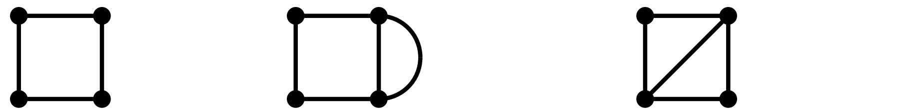


Euler Circuit

It is always possible to make an Euler Circuit or Path if we include *MORE* edges. In fact, doubling all the edges at each vertex will always make an Euler Circuit possible.

Definition 89 (Adding Legal Edges)**.** Adding an edge connection on a graph is **legal** if the edge represents an already existing connection between vertices.

Example 90 (Adding Edges)**.** Starting with the Left Graph, we add a legal edge in the middle and an illegal edge on the right.



Start **Legal Edge Added** **Illegal Edge Added**

When we add legal edges to a graph we want to do it with the fewest possible.

Eulerizing and Semi-Eulerizing

Definition 91 (Eulerizing and Semi-Eulerizing)**.** The process of adding legal edges to a graph in order to:

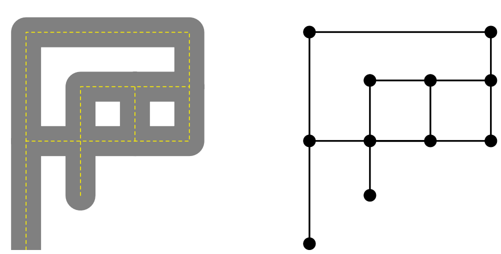
- Make it possible for an Euler Circuit to exist is called **Eulerizing** the graph.
- Make it possible for an Euler Path to exist is called **Semi-Eulerizing** the graph.

If a graph is not already Eulerian (all vertices have even degree) then Eulerizing and Semi-Eulerizing are *ALWAYS* possible!

- For Eulerian, we can Eulerize by doubling *ALL* edges.
- For Semi-Eulerian, we can double *MOST* edges (leaving two vertices with odd degree).

Doubling all/most edges usually won't give us an answer with the fewest added edges.

Example 92 (Snowplow Eulerizing)**.** A snowplow must remove snow from the streets pictured below. How can this done be as efficiently as possible?

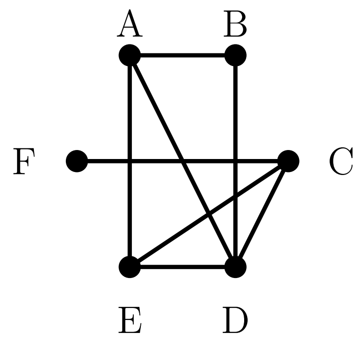


Step 1 Check the degrees of all vertices.
Step 2 Pair up vertices of odd degree. Do this so that the pairs are "close" to one another.
Step 3 Now add edges to connect the pairs. Notice that you may have to go through an intermediate vertex.

Step 4 The final step is to check that you used the fewest number of new edges. Unfortunately, there is no "quick" way to do this.

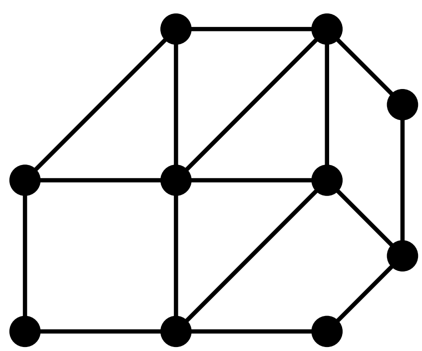
For this example, we needed to add four new edges to Eulerize the graph. The snowplow will have to travel four streets twice.

Example 93 (Eulerize 1)**.** Consider the graph:



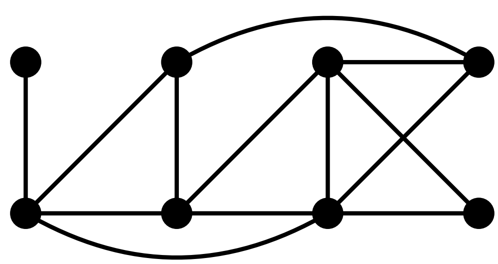
- Find the degrees of the vertices of the graph above!
- How many legal edges do we need to Eulerize the graph?

Example 94 (Eulerize 2)**.** Consider the graph:



- Find the degrees of the vertices of the graph above!
- How many legal edges do we need to Eulerize the graph?

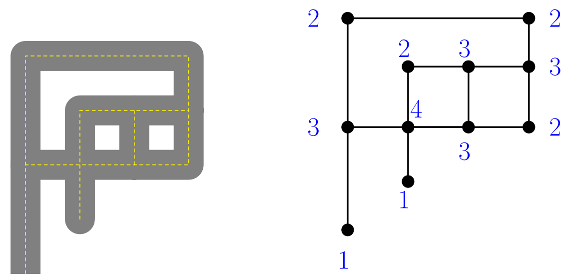
Example 95 (Eulerize 3)**.** Consider the graph:



- Find the degrees of the vertices of the graph above!

- How many legal edges do we need to Eulerize the graph?

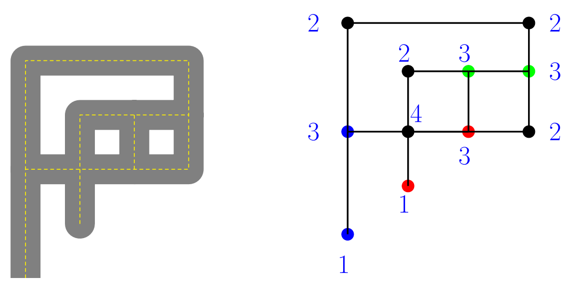
Example 96 (Snowplow Semi-Eulerizing)**.** Here are the degrees of all the vertices.



Step 1 Check the degrees of all vertices.

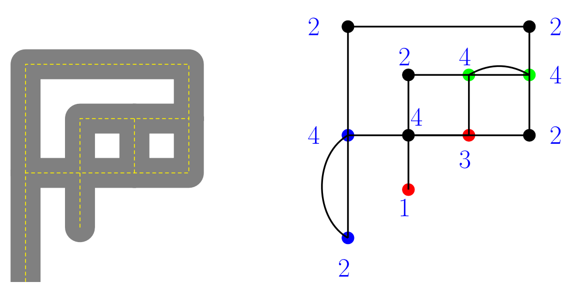
Step 2 Pair up vertices of odd degree. Again, focus on pairs that are "close" to one another.

Each pair is colored differently. You may want to use labels like "P1, P2, etc ..." when you do this on paper.



Step 3 Now add edges to connect *SOME OF THE PAIRS*. Remember, for an Euler Path we need to keep TWO vertices of odd degree.

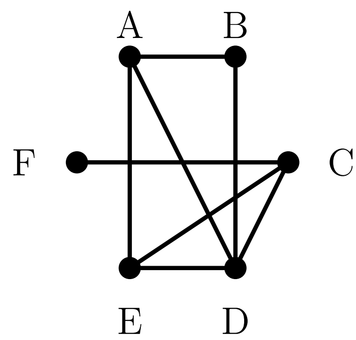
No new connections can be made because these would represent new roads (which don't exist).



Step 4 The final step is to check that you used the fewest number of new edges. Since we have a choice in which vertices of odd degree to leave, we should not make new connections for the red vertices.

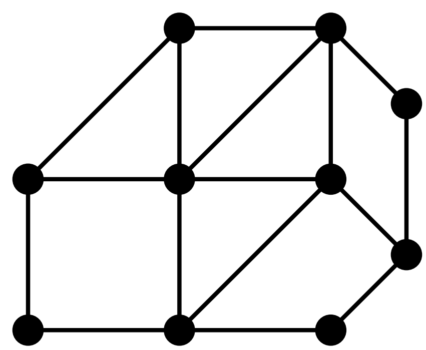
For this example, we needed to add two new edges to Semi-Eulerize the graph.

Example 97 (Semi-Eulerize 1)**.** Consider the graph:



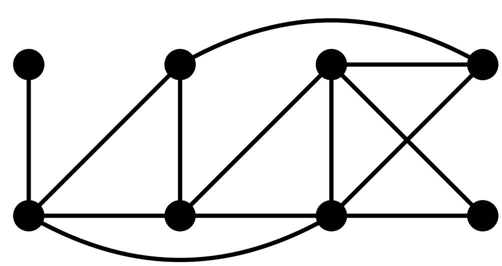
- Find the degrees of the vertices of the graph above!
- How many legal edges do we need to Eulerize the graph?

Example 98 (Semi-Eulerize 2)**.** Consider the graph:



- Find the degrees of the vertices of the graph above!
- How many legal edges do we need to Semi-Eulerize the graph?

Example 99 (Semi-Eulerize 3)**.** Consider the graph:



- Find the degrees of the vertices of the graph above!
- How many legal edges do we need to Semi-Eulerize the graph?